

# Stability of Heat Kernel Estimates for Diffusions with Jumps under Non-local Feynman-Kac Perturbations

Zhen-Qing Chen    and    Lidan Wang

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## Abstract

In this paper we show that the two-sided heat kernel estimates for a class of (not necessarily symmetric) diffusions with jumps are stable under non-local Feynman-Kac perturbations.

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## 1 Introduction

Suppose that  $X$  is a Hunt process on a state space  $E$  with transition semigroup  $\{P_t : t \geq 0\}$ . A Feynman-Kac transform of  $X$  is given by

$$T_t f(x) = \mathbb{E}_x [\text{Exp}(C_t) f(X_t)], \quad (1.1)$$

where  $C_t$  is an additive functional of  $X$  of finite variation. When  $C_t$  is a continuous additive functional of  $X$ , the transform above is called a (local) Feynman-Kac transform. Feynman-Kac transforms play an important role in the probabilistic as well as analytic aspect of potential theory, and also in mathematical physics. For example, Feynman-Kac transforms for Brownian motion on Euclidean spaces have been studied extensively, see [1, 15, 19] and the references therein for a survey on this topic. When  $X$  is discontinuous, additive functionals of  $X$  of finite variation can be discontinuous and there are many of them. When  $C_t$  is a discontinuous additive functional of  $X$ , the transform of (1.1) is called a non-local Feynman-Kac transform. Non-local Feynman-Kac transforms have received quite lot of attention recently in connection with the study of potential theory for discontinuous Markov processes and non-local operators; see, for example, [3, 4, 5, 6, 8, 9, 11, 12, 20] and the references therein. An important question related to Feynman-Kac transforms is the stability of various properties. In particular, stability of heat kernel estimates for purely discontinuous Markov processes under non-local Feynman-Kac perturbations have been studied in [9, 20]. See [2] for a related work. In this paper, we study the stability of heat kernel estimates for diffusions with jumps under non-local Feynman-Kac perturbations.

A generic strong Markov process may have both the continuous (diffusive) part and the purely discontinuous (jumping) part. In Chen and Kumagai [9], symmetric diffusion processes with jumps

on  $\mathbb{R}^d$  having generators

$$\mathcal{L}u(x) = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right) + \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |y-x| > \varepsilon\}} (u(y) - u(x)) \frac{c(x,y)}{|x-y|^{d+\alpha}} dy \quad (1.2)$$

are studied, where  $\alpha \in (0, 2)$ ,  $A(x) = (a_{ij}(x))_{1 \leq i,j \leq d}$  is a measurable  $d \times d$  matrix-valued function on  $\mathbb{R}^d$  that is uniformly elliptic and bounded, and  $c(x, y)$  is symmetric function that is bounded between two positive constants. It is shown that there is a Feller process  $X$  having strong Feller property associated with  $\mathcal{L}$ , which we call symmetrid diffusion with jumps. The Feller process  $X_t$  has a jointly Hölder continuous transition density function  $p(t, x, y)$  with respect to the Lebesgue measure on  $\mathbb{R}^d$  and the following two-sided estimates hold. There exist positive constants  $c_k$ ,  $1 \leq k \leq 4$ , such that for every  $t > 0$  and  $x, y \in \mathbb{R}^d$ ,

$$\begin{aligned} & c_1 \left( t^{-d/2} \wedge t^{-d/\alpha} \right) \wedge \left( t^{-d/2} \text{Exp} \left( -\frac{c_2|x-y|^2}{t} \right) + t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \\ & \leq p(t, x, y) \leq c_3 \left( t^{-d/2} \wedge t^{-d/\alpha} \right) \wedge \left( t^{-d/2} \text{Exp} \left( -\frac{c_4|x-y|^2}{t} \right) + t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right). \end{aligned} \quad (1.3)$$

For  $a, b \in \mathbb{R}$ ,  $a \wedge b := \min\{a, b\}$  and  $a \vee b := \max\{a, b\}$ .

Recently the following non-symmetric non-local operator

$$\mathcal{L}u(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(x) \frac{\partial u(x)}{\partial x_j} + \int_{\mathbb{R}^d} (u(x+z) - u(x) - 1_{\{|z| \leq 1\}} \cdot \nabla u(x)) \frac{c(x, z)}{|z|^{d+\alpha}} dz, \quad (1.4)$$

has been studied in [7], where  $\alpha \in (0, 2)$ , and  $A = (a_{ij}(x))_{1 \leq i,j \leq d}$  is a measurable  $d \times d$  matrix-valued function on  $\mathbb{R}^d$  that is uniformly elliptic and bounded, and is Hölder continuous,  $b(x) = (b_1(x), \dots, b_d(x))$  is an  $\mathbb{R}^d$ -valued function that is in some Kato class, and  $c(x, z) \geq 0$  is a bounded measurable function such that when  $\alpha = 1$ ,

$$\int_{\{r < |z| < R\}} z c(x, z) dz = 0 \quad \text{for every } 0 < r < R < \infty.$$

Chen, Hu, Xie and Zhang [7] showed, among other things, that there is a Feller process  $X_t$  having strong Feller property associated with the above generator, and  $X_t$  has a jointly continuous transition density function  $p(t, x, y)$  with respect to the Lebesgue measure on  $\mathbb{R}^d$ . Moreover, when  $c(x, z)$  is bounded between two positive constants, the two-sided estimates (1.3) are shown to hold for  $p(t, x, y)$  on  $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$  for every  $T > 0$ . In fact, more general time-dependent operators of the form (1.4) are studied in [7].

In this paper, we start with a Hunt process  $X$  on  $\mathbb{R}^d$  with  $d \geq 2$  that has a jointly continuous transition density function  $p(t, x, y)$  that enjoys two-sided estimates (1.3) on  $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ . Under this assumption, the Hunt process  $X$  has a Lévy system  $(N(x, dy), dt)$  with  $N(x, dy) = \frac{c(x, y)}{|x-y|^{d+\alpha}} dy$  for some measurable function  $c(x, y)$  bounded between two positive constants; see (3.7) below. That is, for every non-negative function  $\varphi(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$  that vanishes along the diagonal,

$$\mathbb{E}_x \left[ \sum_{0 < s \leq t} \varphi(X_{s-}, X_s) \right] = \mathbb{E}_x \left[ \int_0^t \int_{\mathbb{R}^d} \varphi(X_s, y) N(X_s, dy) ds \right], \quad x \in \mathbb{R}^d, \quad t > 0.$$

Here we use the convention that we extend the definition of functions to cemetery point  $\partial$  by setting 0 value there; for example  $\varphi(x, \partial) = 0$ . For convenience, we take  $T = 1$ . We will study the stability of heat kernel estimates under non-local Feynman-Kac transform:

$$T_t f(x) = \mathbb{E}_x \left[ \exp \left( A_t^\mu + \sum_{s \leq t} F(X_{s-}, X_s) \right) f(X_t) \right],$$

where  $A^\mu$  is a continuous additive functional of  $X$  of finite variations having signed Revuz measure  $\mu$  and  $F(x, y)$  is a bounded measurable function vanishing on the diagonals. We point out that in this paper we do not require  $F$  to be symmetric. Informally, the semigroup  $(T_t^{\mu, F}; t \geq 0)$  has generator

$$\mathcal{A}f(x) = (\mathcal{L} + \mu)f(x) + \int_{\mathbb{R}^d} \left( e^{F(x, y)} - 1 \right) f(y) N(x, dy), \quad (1.5)$$

where  $\mathcal{L}$  is the infinitesimal generator of  $X$ ; see [11, Remark 1 on p.53] for a calculation. We show that if  $\mu$  and  $F$  are in certain Kato class of  $X$ , the non-local Feynman-Kac semigroup  $\{T_t; t \geq 0\}$  has a heat kernel  $q(t, x, y)$  and  $q(t, x, y)$  has two-sided estimates (1.3) on  $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$  but with a set of possible different constants  $c_k$ ,  $1 \leq k \leq 4$ . Comparing with [9, 20, 2], the novelty of this paper is that  $X$  has both the diffusive and jumping components, and that the Gaussian bounds in (1.3) have different constants  $c_2$  and  $c_4$  in the exponents for the upper and lower bound estimates. These features made the perturbation estimates more challenging.

The rest of the paper is organized as follows. Section 2 gives the basic setup of the problem and the statement of the main results of this paper. In Section 3, we various 3P type inequalities needed to study non-local Feynman-Kac perturbations. Proof of the main results, the two-sided estimates for the heat kernel of the Feynman-Kac semigroup, is given in Section 4.

In this paper, we adopt the following notations. We use “:=” as a way of definition. For two positive functions  $f$  and  $g$ , notation  $f \asymp g$  means that there is a constant  $c \geq 1$  so that  $g/c \leq f \leq cg$ , while notation  $f \lesssim g$  (respectively,  $f \gtrsim g$ ) means there is a constant  $c > 0$  so that  $f \leq cg$  (respectively,  $f \geq cg$ ).

## 2 Preliminaries and Main Result

Throughout the remainder of this paper, we assume that  $X$  is a Hunt process on  $\mathbb{R}^d$  with  $d \geq 2$  having a jointly continuous transition density function  $p(t, x, y)$  with respect to the Lebesgue measure on  $\mathbb{R}^d$  and that the two-sided estimates (1.3) holds for  $p(t, x, y)$  on  $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ . Since we are concerned with heat kernel estimates of (1.3) on fixed time intervals, it is desirable to rewrite the estimates in the following equivalent but more compact form. This equivalent form (2.1) is given in [7]. For reader's convenience, we give a proof here.

**Lemma 2.1.** *Two-sided estimates (1.3) for  $p(t, x, y)$  on  $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$  is equivalent to the following. There exist constants  $C \geq 1$  and  $\lambda \in (0, 1]$  such that for  $0 < t \leq 1$  and  $x, y \in \mathbb{R}^d$ ,*

$$C^{-1} (\Gamma_{c_2}(t; x - y) + \eta(t; x - y)) \leq p(t, x, y) \leq C (\Gamma_{c_4}(t; x - y) + \eta(t; x - y)), \quad (2.1)$$

where

$$\Gamma_\lambda(t; x) := t^{-d/2} e^{-\lambda|x|^2/t} \quad \text{and} \quad \eta(t; x) := \frac{t}{(t^{1/2} + |x|)^{d+\alpha}}. \quad (2.2)$$

**Proof.** Note that  $t^{-d/2} \leq t^{-d/\alpha}$  for  $t \in (0, 1]$ , and

$$\frac{1}{2}(a \wedge b + a \wedge c) \leq a \wedge (b + c) \leq a \wedge b + a \wedge c \quad \text{for } a, b, c > 0.$$

Thus for  $\lambda > 0$  and  $t \in (0, 1]$ ,

$$\begin{aligned}
& \left( t^{-d/2} \wedge t^{-d/\alpha} \right) \wedge \left( t^{-d/2} \text{Exp} \left( -\frac{\lambda r^2}{t} \right) + t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}} \right) \\
& \asymp t^{-d/2} \text{Exp} \left( -\frac{\lambda r^2}{t} \right) + t^{-d/2} \wedge t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}} \\
& \asymp \begin{cases} t^{-d/2} \text{Exp} \left( -\frac{\lambda r^2}{t} \right) + t^{-d/2} \wedge \frac{t}{r^{d+\alpha}} \asymp t^{-d/2} \text{Exp} \left( -\frac{\lambda r^2}{t} \right) + \frac{t}{(t^{1/2}+r)^{d+\alpha}} & \text{if } r \geq t^{1/2} (\geq t^{1/\alpha}), \\ t^{-d/2} + t^{-d/2} \wedge \frac{t}{r^{d+\alpha}} \asymp t^{-d/2} & \text{if } t^{1/\alpha} < r \leq t^{1/2}, \\ t^{-d/2} + t^{-d/2} \wedge t^{-d/\alpha} \asymp t^{-d/2} & \text{if } 0 < r < t^{1/\alpha} (\leq t^{1/2}) \end{cases} \\
& \asymp t^{-d/2} \text{Exp} \left( -\frac{\lambda r^2}{t} \right) + \frac{t}{(t^{1/2}+r)^{d+\alpha}},
\end{aligned}$$

where the last line is due to the fact that for  $0 < r \leq t^{1/2}$ ,  $\frac{t}{(t^{1/2}+r)^{d+\alpha}} \leq t^{1-(d+\alpha)/2} \leq t^{-d/2}$ . This establishes the lemma.  $\square$

We now introduce some Kato classes for signed measures and for functions  $F$  used in non-local Feynman-Kac perturbation. For a  $\sigma$ -finite signed measure  $\mu$ , we use  $\mu^+$  and  $\mu^-$  to denote its positive and negative part in its Jordan decomposition, and its total variation measure is given by  $|\mu| := \mu^+ + \mu^-$ . For a signed measure  $\mu$  on  $\mathbb{R}^d$ , using the notations in (2.2), we define

$$N_\mu^{\alpha, \lambda} := \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} (\Gamma_\lambda(t; x-y) + \eta(t; x-y)) |\mu|(dy) ds. \quad (2.3)$$

For a function  $F(x, y)$  defined on  $\mathbb{R}^d \times \mathbb{R}^d$  that vanishes along the diagonal, we define

$$N_F^{\alpha, \lambda}(t) := \sup_{y \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (\Gamma_\lambda(t; y-z) + \eta(t; y-z)) \frac{|F(z, w)| + |F(w, z)|}{|z-w|^{d+\alpha}} dw dz ds. \quad (2.4)$$

**Definition 2.2.** (i) A signed measure  $\mu$  on  $\mathbb{R}^d$  is said to be in the Kato class  $\mathbf{K}_\alpha$  if  $\lim_{t \downarrow 0} N_\mu^{\alpha, \lambda}(t) = 0$  for some and hence for all  $\lambda > 0$ . A measurable function  $f$  on  $\mathbb{R}^d$  is said to be in Kato class  $\mathbf{K}_\alpha$  if  $|f(x)|\mu(dx) \in \mathbf{K}_\alpha$ .

(ii) A bounded measurable function  $F$  on  $\mathbb{R}^d \times \mathbb{R}^d$  vanishing on the diagonal, is said to be Kato class  $\mathbf{J}_\alpha$  if  $\lim_{t \downarrow 0} N_F^{\alpha, \lambda}(t) = 0$  for some and hence for all  $\lambda > 0$ .

Clearly, if  $F, G \in \mathbf{J}_\alpha$  and  $a \in \mathbb{R}$ , then so are  $aF$ ,  $e^F - 1$ ,  $F + G$  and  $FG$ . By Hölder inequality, it is easy to check that  $L^\infty(\mathbb{R}^d) + L^p(\mathbb{R}^d) \subset \mathbf{K}_\alpha$  for every  $p > d/2$ . For  $\mu \in \mathbf{K}_\alpha$  and  $F \in \mathbf{J}_\alpha$ , we can define an additive function of  $X$  by

$$A_t^{\mu, F} = A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s),$$

where  $A_t^\mu$  is a continuous additive functional of  $X$  having  $\mu$  as its Revuz measure. It is easy to check that  $A_t^{\mu, F}$  is well defined and is of finite variations on compact time intervals. We can then define the following non-local Feynman-Kac semigroup of  $X$  by

$$T_t^{\mu, F} f(x) = \mathbb{E}_x \left[ \text{Exp}(A_t^{\mu, F}) f(X_t) \right], \quad t \geq 0, x \in \mathbb{R}^d. \quad (2.5)$$

The goal of this paper is to study the stability of heat kernel estimates under the above non-local Feynman-Kac transform.

**Theorem 2.3.** Suppose  $X$  is a Hunt process on  $\mathbb{R}^d$  that has a jointly continuous transition density function  $p(t, x, y)$  with respect to the Lebesgue measure and that the two-sided heat kernel estimates (1.3) holds for  $p(t, x, y)$  on  $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ . Let  $\mu \in \mathbf{K}_\alpha$  and  $F(x, y)$  be a measurable function so that  $F_1 := e^F - 1 \in \mathbf{J}_\alpha$ . Then the non-local Feynman-Kac semigroup  $(T_t^{\mu, F}; t \geq 0)$  has a jointly continuous kernel  $q(t, x, y)$  so that  $T_t^{\mu, F} f(x) = \int_{\mathbb{R}^d} q(t, x, y) f(y) dy$  for every bounded Borel measurable function  $f$  on  $\mathbb{R}^d$ . Moreover, there exist positive constants  $c_1$  and  $K$  that depend on  $(d, \alpha, C, N_{\alpha, F}^{\alpha, c_4}, \|F_1\|_\infty)$  so that for any  $t > 0$  and  $x, y \in \mathbb{R}^d$ ,

$$q(t, x, y) \leq c_1 e^{Kt} (\Gamma_{2c_4/3}(t; x - y) + \eta(t; x - y)).$$

Here  $c_4$  is the constant in (1.3). If in addition,  $F \in \mathbf{J}_\alpha$ , then there exist positive constants  $c_2, \lambda_1$  and  $K_1$  that depend on  $(d, \alpha, C, N_{\alpha, F}^{\alpha, c_4}, \|F\|_\infty)$  so that for any  $t > 0$  and  $x, y \in \mathbb{R}^d$ ,

$$q(t, x, y) \geq c_2 e^{-K_1 t} (\Gamma_{\lambda_1}(t; x - y) + \eta(t; x - y)).$$

### 3 3P inequalities

In this section we will establish various 3P type inequalities, which are key ingredients in the proof of Theorem 2.3. Lemma 3.1, Lemma 3.3, Lemma 3.4, and Lemma 3.5 are dealing with  $\Gamma_c(t; x - y)$  and  $\eta(t; x - y)$  as defined in (2.2). Theorem 3.2 and Theorem 3.6 are the main results of this section.

**Lemma 3.1.** For  $0 < s < t$ , and  $x, y, z \in \mathbb{R}^d$ ,

(i) There exists a constant  $C_1 = C_1(d, \alpha)$  such that

$$\eta(t - s; x - z) \eta(s; z - y) \leq C_1 \eta(t; x - y) (\eta(t - s; x - z) + \eta(s; z - y)). \quad (3.1)$$

(ii) For  $0 < a < b$ , there exists a constant  $C_2 = C_2(d, a, b)$  such that for any measure  $\mu$  on  $\mathbb{R}^d$ ,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \Gamma_a(t - s; x - z) \Gamma_b(s; z - y) |\mu|(dz) ds \\ & \leq C_2 \Gamma_a(t; x - y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_c(s; x - y) |\mu|(dy) ds, \end{aligned} \quad (3.2)$$

where  $c = (b - a) \wedge \frac{a}{2}$ .

(iii) There exists a constant  $C_3 = C_3(d, \alpha, a)$  such that for any measure  $\mu$  on  $\mathbb{R}^d$ ,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \Gamma_a(t - s; x - z) \eta(s; z - y) |\mu|(dz) ds \\ & \leq C_3 \left( \Gamma_a(t; x - y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \eta(s; x - y) |\mu|(dy) ds \right. \\ & \quad \left. + \eta(t; x - y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_a(s; x - y) |\mu|(dy) ds \right). \end{aligned} \quad (3.3)$$

**Proof.** (i) For  $0 < s < t$  and  $x, y, z \in \mathbb{R}^d$ , we have

$$\begin{aligned}
& \frac{\eta(t-s; x-z)\eta(s; z-y)}{\eta(t; x-y)} \\
&= \frac{(t-s)s}{t} \left( \frac{t^{1/2} + |x-y|}{((t-s)^{1/2} + |x-z|)(s^{1/2} + |z-y|)} \right)^{d+\alpha} \\
&\leq ((t-s) \wedge s) \left( \frac{((t-s) + s)^{1/2} + |x-z| + |z-y|}{((t-s)^{1/2} + |x-z|)(s^{1/2} + |z-y|)} \right)^{d+\alpha} \\
&\leq 2^{d+\alpha} ((t-s) \wedge s) \left( \frac{1}{((t-s)^{1/2} + |x-z|)^{d+\alpha}} + \frac{1}{(s^{1/2} + |z-y|)^{d+\alpha}} \right) \\
&\leq 2^{d+\alpha} (\eta(t-s; x-z) + \eta(s; z-y)).
\end{aligned}$$

(ii) The proof for this part is similar to that for Lemma 3.1 in [21]. We first write

$$\begin{aligned}
J(t, x, y) &:= \int_0^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\
&= \int_0^{\rho t} \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\
&\quad + \int_{\rho t}^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds
\end{aligned}$$

Applying the elementary inequality

$$\frac{|x-z|^2}{t-s} + \frac{|z-y|^2}{s} \geq \frac{|x-y|^2}{t}, \text{ for } 0 < s < t, \quad (3.4)$$

one can obtain

$$\begin{aligned}
& \int_0^{\rho t} \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\
&= \int_0^{\rho t} \int_{\mathbb{R}^d} (t-s)^{-d/2} s^{-d/2} \text{Exp}\left(-a \frac{|x-z|^2}{t-s}\right) \text{Exp}\left(-b \frac{|z-y|^2}{s}\right) |\mu|(dz) ds \\
&= \int_0^{\rho t} \int_{\mathbb{R}^d} (t-s)^{-d/2} s^{-d/2} \text{Exp}\left(-a \left(\frac{|x-z|^2}{t-s} + \frac{|z-y|^2}{s}\right)\right) \text{Exp}\left(-(b-a) \frac{|z-y|^2}{s}\right) |\mu|(dz) ds \\
&\leq \int_0^{\rho t} \int_{\mathbb{R}^d} (t-s)^{-d/2} s^{-d/2} \text{Exp}\left(-a \frac{|x-y|^2}{t}\right) \text{Exp}\left(-(b-a) \frac{|z-y|^2}{s}\right) |\mu|(dz) ds \\
&\leq (1-\rho)^{-d/2} \Gamma_a(t; x-y) \int_0^{\rho t} \int_{\mathbb{R}^d} \Gamma_{b-a}(s; z-y) |\mu|(dz) ds.
\end{aligned}$$

For the other term, by defining  $U = \{|z-y| \geq |x-y|(a/b)^{1/2}\}$ , we have

$$\begin{aligned}
& \int_{\rho t}^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\
&= \int_{\rho t}^t \int_U \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds + \int_{\rho t}^t \int_{U^c} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\
&\leq (\rho t)^{-d/2} \text{Exp}\left(-\frac{a|x-y|^2}{t}\right) \int_{\rho t}^t \int_U (t-s)^{-d/2} \text{Exp}\left(-\frac{a|x-z|^2}{t-s}\right) |\mu|(dz) ds \\
&\quad + (\rho t)^{-d/2} \int_{\rho t}^t \int_{U^c} (t-s)^{-d/2} \text{Exp}\left(-\frac{a|x-z|^2}{t-s}\right) |\mu|(dz) ds.
\end{aligned}$$

On  $U^c$ , we have the inequality,

$$|x - z| \geq |x - y| - |y - z| \geq |x - y|(1 - (a/b)^{1/2}),$$

thus,

$$\begin{aligned} & \int_{\rho t}^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\ & \leq \rho^{-d/2} \Gamma_a(t; x-y) \int_{\rho t}^t \Gamma_a(t-s; x-z) |\mu|(dz) ds \\ & \quad + (\rho t)^{-d/2} \int_{\rho t}^t \int_{U^c} (t-s)^{-d/2} \text{Exp}\left(-\frac{a|x-z|^2}{2(t-s)}\right) \text{Exp}\left(-\frac{a(1-(a/b)^{1/2})^2|x-y|^2}{2(t-s)}\right) |\mu|(dz) ds \\ & \leq \rho^{-d/2} \Gamma_a(t; x-y) \int_{\rho t}^t \Gamma_a(t-s; x-z) |\mu|(dz) ds \\ & \quad + (\rho t)^{-d/2} \text{Exp}\left(-\frac{a(1-(a/b)^{1/2})^2|x-y|^2}{2(1-\rho)t}\right) \int_{\rho t}^t \int_{U^c} (t-s)^{-d/2} \text{Exp}\left(-\frac{a|x-z|^2}{2(t-s)}\right) |\mu|(dz) ds, \end{aligned}$$

by selecting  $\rho$  such that  $2(1-\rho) = (1-(a/b)^{1/2})^2$ , we would achieve the estimate in (3.2), with  $c = (b-a) \wedge \frac{a}{2}$ , and  $C_2$  depends on  $d, a, b$ .

(iii) For  $0 < s < t$ , if  $|x-y| \leq t^{1/2}$ , we have

$$\begin{aligned} \Gamma_a(t-s; x-z) & \leq 2^{d/2} t^{-d/2} \leq 2^{d/2} e^a \Gamma_a(t; x-y), & \text{for } s \in (0, t/2]; \\ \eta(s; z-y) & \leq 4^{d+\alpha} \eta(t; x-y), & \text{for } s \in (t/2, t). \end{aligned}$$

If  $|x-y| > t^{1/2}$ , consider on  $V = \{|y-z| \geq |x-y|/2\}$ , we would have  $\eta(s; z-y) \leq 2^{d+\alpha} \eta(t; x-y)$  for all  $0 < s < t$ .

On  $V^c$ ,  $|x-z| \geq |x-y| - |y-z| \geq |x-y|/2$ , we would have  $\Gamma_a(t-s; x-z) \leq \gamma \Gamma_a(t; x-y)$ , where  $\gamma$  depends on  $a, d$ .

The estimate (3.3) directly follows from the above discussion.  $\square$

Recall the definition of  $N_\mu^{\alpha, \lambda}$  from (2.3). We next derive an integral 3P type inequality for  $p(t, x, y)$  in small time, by using two-sided heat kernel estimates in Lemma 2.1. For notational convenience, let  $\lambda = c_4$ , where  $c_4$  is the positive constant in (1.3).

**Theorem 3.2.** *For any  $\mu \in \mathbf{K}_\alpha$ , and any  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ ,*

$$\int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) ds \leq M_1 p_{2\lambda/3}(t, x, y) N_\mu^{\alpha, \lambda/3}(t), \quad (3.5)$$

where  $M_1$  depends on  $d, \alpha, C, \lambda$ , and  $p_{2\lambda/3}(t, x, y) := \Gamma_{2\lambda/3}(t; x-y) + \eta(t; x-y)$ .

**Proof.** By Lemma 2.1, we have

$$p(t, x, y) \leq C (\Gamma_\lambda(t; x-y) + \eta(t; x-y)) \quad \text{for } t \in (0, 1].$$

Thus for  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ ,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) ds \\ & \leq C \left( \int_0^t \int_{\mathbb{R}^d} \Gamma_\lambda(t-s; x-z) \Gamma_{2\lambda/3}(s; z-y) |\mu|(dy) ds + \int_0^t \int_{\mathbb{R}^d} \eta(t-s; x-z) \eta(s; z-y) |\mu|(dz) ds \right. \\ & \quad \left. + \int_0^t \int_{\mathbb{R}^d} \Gamma_\lambda(t-s; x-z) \eta(s; z-y) |\mu|(dz) ds + \int_0^t \int_{\mathbb{R}^d} \eta(t-s; x-z) \Gamma_{2\lambda/3}(s; z-y) |\mu|(dz) ds \right) \end{aligned}$$

Applying Lemma 3.1, we would have

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) ds \\
& \leq C \left( C_2 \Gamma_{2\lambda/3}(t, x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_{\lambda/3}(s; x-y) |\mu|(dy) ds \right. \\
& \quad + 2C_1 \eta(t; x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \eta(s; x-y) |\mu|(dy) ds \\
& \quad + 2C_3 \Gamma_{2\lambda/3}(t; x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \eta(s; x-y) |\mu|(dy) ds \\
& \quad \left. + 2C_3 \eta(t; x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(s; x-y) |\mu|(dy) ds \right),
\end{aligned}$$

where  $C_1, C_2, C_3$  depend on  $d, \alpha, C, \lambda$ . Altogether, let  $M_1 = C(2C_1 \vee C_2 \vee 2C_3)$ , we have

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) ds \\
& \leq M_1 (\Gamma_{2\lambda/3}(t, x-y) + \eta(t; x-y)) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} (\Gamma_{\lambda/3}(s; x-y) + \eta(s; x-y)) |\mu|(dy) ds \\
& = M_1 p_{2\lambda/3}(t, x, y) N_\mu^{\alpha, \lambda/3}(t).
\end{aligned}$$

□

We will use the following notations: for any  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ ,

$$\begin{aligned}
V_{x,y} &:= \left\{ (z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \geq 4(|y-w| \wedge |x-z|) \right\}; \\
U_{x,y} &:= V_{x,y}^c.
\end{aligned}$$

First, similar as the discussion in [9] (see Theorem 2.7), we could have the generalized integral 3P inequality for  $\eta(t; x-y)$ .

**Lemma 3.3.** *There exists a constant  $C_4 = C_4(\alpha, d)$  such that for any non-negative bounded function  $F(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$ , the followings are true for  $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ .*

(i) *If  $|x-y| \leq t^{1/2}$ , then*

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\
& \leq C_4 \eta(t; x-y) \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (\eta(s; x-z) + \eta(s; w-y)) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds.
\end{aligned}$$

(ii) *If  $|x-y| > t^{1/2}$ , then*

$$\begin{aligned}
& \int_0^t \int_{U_{x,y}} \eta(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\
& \leq C_4 \eta(t; x-y) \int_0^t \int_{U_{x,y}} (\eta(s; x-z) + \eta(s; w-y)) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds.
\end{aligned}$$



(iii) If  $|x - y| > t^{1/2}$ , then

$$\int_0^t \int_{V_{x,y}} \eta(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \leq C_4 \|F\|_\infty \eta(t; x-y),$$

where  $\|F\|_\infty$  denotes the  $L^\infty$ -norm of  $F$  on  $\mathbb{R}^d \times \mathbb{R}^d$ .

Now we proceed to get the generalized integral 3P inequality for  $\Gamma_c(t; x-y)$ .

**Lemma 3.4.** For  $0 < a < b$ , there exists a constant  $C_5 = C_5(a, b, d)$  such that for any non-negative bounded function  $F(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$ , the followings are true for  $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ .

(i) If  $|x - y| \leq t^{1/2}$ , then

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq C_5 \Gamma_b(t; x-y) \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (\Gamma_a(s; x-z) + \Gamma_b(s; w-y)) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds. \end{aligned}$$

(ii) If  $|x - y| > t^{1/2}$ , then

$$\begin{aligned} & \int_0^t \int_{U_{x,y}} \Gamma_a(t-s; x-z) \Gamma_b(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq C_5 \Gamma_a(t; x-y) \int_0^t \int_{U_{x,y}} (\Gamma_a(s; x-z) + \Gamma_b(s; w-y)) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds. \end{aligned}$$

(iii) If  $|x - y| > t^{1/2}$ , then

$$\int_0^t \int_{V_{x,y}} \Gamma_a(t-s; x-z) \Gamma_b(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \leq C_5 \|F\|_\infty \eta(t; x-y).$$

**Proof.** (i) If  $|x - y| \leq t^{1/2}$ , we have  $\Gamma_a(t-s; x-z) \leq 2^{d/2} e^b \Gamma_b(t; x-y)$  when  $s \in (0, t/2]$ ;  $\Gamma_b(s; w-y) \leq 2^{d/2} e^b \Gamma_b(t; x-y)$  when  $s \in (t/2, t)$ . Then (i) follows naturally.

(ii) If  $|x - y| > t^{1/2}$ , we let

$$U_1 := \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |y-w| > 4^{-1}|x-y|, |y-w| \geq |x-z|\};$$

$$U_2 := \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |x-z| > 4^{-1}|x-y|\}.$$

Note that  $\Gamma_b(s; w-y) \leq \gamma_1 \Gamma_b(t; x-y)$  on  $U_1$  for  $s \in (0, t)$ ;  $\Gamma_a(t-s; x-z) \leq \gamma_2 \Gamma_a(t; x-y)$  on  $U_2$  for  $s \in (0, t)$ , where  $\gamma_1 := \gamma_1(b, d)$  and  $\gamma_2 := \gamma_2(a, d)$ . Since  $U_{x,y} = U_1 \cup U_2$ , (ii) follows directly.

(iii) On  $V_{x,y}$ ,  $|z-w| \geq 2^{-1}|x-y|$ . Hence

$$\begin{aligned} & \int_0^t \int_{V_{x,y}} \Gamma_a(t-s; x-z) \Gamma_b(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq 2^{d+\alpha} |x-y|^{-(d+\alpha)} \|F\|_\infty \int_0^t \int_{V_{x,y}} \Gamma_a(t-s; x-z) \Gamma_b(s; w-y) dz dw ds \\ & \lesssim \frac{1}{t} \eta(t; x-y) \|F\|_\infty \int_0^t \left( \int_{\mathbb{R}^d} \Gamma_a(t-s; z) dz \right) \left( \int_{\mathbb{R}^d} \Gamma_b(s; w) dw \right) ds \\ & \lesssim \eta(t; x-y) \|F\|_\infty. \end{aligned}$$

This completes the proof of the lemma.  $\square$

We next establish a generalized integral 3P inequality involving both  $\Gamma_c(t; x-y)$  and  $\eta(t; x-y)$ .

**Lemma 3.5.** *There exists a constant  $C_6 = C_6(c, \alpha, d)$  such that for any non-negative bounded function  $F(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$ , the followings are true for  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ .*

(i) *If  $|x - y| \leq t^{1/2}$ , then*

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq C_6 \left( \Gamma_c(t; x-y) \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \right. \\ & \quad \left. + \eta(t; x-y) \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_c(s; x-z) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \right). \end{aligned}$$

(ii) *If  $|x - y| > t^{1/2}$ , then*

$$\begin{aligned} & \int_0^t \int_{U_{x,y}} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq C_6 \left( \Gamma_c(t; x-y) \int_0^t \int_{U_{x,y}} \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \right. \\ & \quad \left. + \eta(t; x-y) \int_0^t \int_{U_{x,y}} \Gamma_c(s; x-z) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \right). \end{aligned}$$

(iii) *If  $|x - y| > t^{1/2}$ , then*

$$\int_0^t \int_{V_{x,y}} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \leq C_6 \|F\|_\infty \eta(t; x-y).$$

**Proof.** (i) If  $|x - y| \leq t^{1/2}$ , we have  $\Gamma_c(t-s; x-z) \leq 2^{d/2} e^c \Gamma_c(t; x-y)$  when  $s \in (0, t/2]$ ;  $\eta(s; w-y) \leq 4^{d+\alpha} \eta(t; x-y)$  when  $s \in (t/2, t]$ . Thus, we have (i) hold naturally.

(ii) If  $|x - y| > t^{1/2}$ , we continue to use the decomposition  $U_{x,y} = U_1 \cup U_2$  in the proof of Lemma 3.4, and observe that  $\Gamma_c(t-s; x-z) \leq \gamma_4(t; x-y)$  on  $U_2$  for  $s \in (0, t)$ , where  $\gamma_4$  depends on  $d, c$ . Thus, we first have

$$\begin{aligned} & \int_0^t \int_{U_2} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq \gamma_4 \Gamma_c(t; x-y) \int_0^t \int_{U_2} \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds. \end{aligned}$$

Also, observe that  $\eta(s; w-y) \leq 4^{d+\alpha} \eta(t; x-y)$  when  $s \in (0, t)$  and  $(z, w) \in U_1$ . Thus,

$$\begin{aligned} & \int_0^t \int_{U_1} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq 4^{d+\alpha} \eta(t; x-y) \int_0^t \int_{U_1} \Gamma_c(s; x-z) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds. \end{aligned}$$

Altogether, (ii) holds directly.

(iii) Note that on  $V_{x,y}$ ,  $|z-w| \geq 2^{-1}|x-y|$ . Also, there exists  $\gamma_5$  depending on  $d, c, \alpha$  such that  $\int_{\mathbb{R}^d} \Gamma_c(t; x-y) dy \leq \gamma_5$  and  $\int_{\mathbb{R}^d} \eta(t; x-y) dy \leq \gamma_5 t^{(2-\alpha)/2}$ . Thus,

$$\begin{aligned} & \int_0^t \int_{V_{x,y}} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq 2^{d+\alpha} |x-y|^{-(d+\alpha)} \|F\|_\infty \int_0^t \gamma_5^2 s^{(2-\alpha)/2} ds \\ & \leq 2^{d+\alpha} \gamma_5^2 \|F\|_\infty |x-y|^{-(d+\alpha)} t^{(4-\alpha)/2}, \end{aligned}$$

for  $t \leq 1$  and  $|x-y| > t^{1/2}$ , there exists  $C_6$  depending on  $d, \alpha, c$  such that

$$\int_0^t \int_{V_{x,y}} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \leq C_6 \|F\|_\infty \eta(t; x-y).$$

□

Recall that the definition of  $N_F^{\alpha, \lambda}(t)$  from (2.4). Note that a Hunt process  $X_t$  admits a Lévy system  $(N(x, dy), H_t)$ , where  $N(x, dy)$  is a kernel and  $H_t$  is a positive continuous additive functional of  $X_t$ ; that is, for any  $x \in \mathbb{R}^d$ , any stopping time  $T$  and any non-negative measurable function  $f$  on  $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ , vanishing on the diagonal,

$$\mathbb{E}_x \left[ \sum_{s \leq T} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[ \int_0^T \int_{\mathbb{R}^d} f(s, X_s, y) N(X_s, dy) dH_s \right]. \quad (3.6)$$

Since  $X_t$  has transition density function  $p(t, x, y)$  with respect to the Lebesgue measure, it follows that the Revuz measure  $\mu_H$  of  $H$  is absolutely continuous with respect to the Lebesgue measure. So we can take  $\mu_H(dx) = dx$ , in other words, we can take  $H_t \equiv t$ . By two-sided heat kernel estimates (1.3) for the Hunt process  $X$  and the fact that  $N(x, dy)$  is the weak limit of  $p(t, x, y) dy / t$  as  $t \rightarrow 0$ , we have

$$H_t = t \quad \text{and} \quad N(x, dy) = \frac{c(x, y)}{|x-y|^{d+\alpha}} dy \quad (3.7)$$

for some measurable function  $c(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$  that is bounded between two positive constants.

**Theorem 3.6.** *Suppose  $F(x, y)$  is a measurable function so that  $F_1 = e^F - 1 \in \mathbf{J}_\alpha$ . There is a constant  $M_2 > 0$  so that for any  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ ,*

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq M_2 p_{2\lambda/3}(t, x, y) \left( N_{F_1}^{\alpha, \lambda/3}(t) + \|F_1\|_\infty \mathbf{1}_{\{|x-y| > t^{1/2}\}} \right), \end{aligned} \quad (3.8)$$

In particular, on  $U_{x,y} = \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \geq 4(|y-w| \wedge |x-z|)\}^c$ ,

$$\int_0^t \int_{U_{x,y}} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \leq M_2 p_{2\lambda/3}(t, x, y) N_{F_1}^{\alpha, \lambda/3}(t). \quad (3.9)$$

**Proof.** By Lemma 2.1,

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\
& \leq C \left( \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_\lambda(t-s; x-z) \Gamma_{2\lambda/3}(s; w-y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \right. \\
& \quad + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta(t-s; x-z) \eta(s; w-y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\
& \quad + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_\lambda(t-s; x-z) \eta(s; w-y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\
& \quad \left. + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta(t-s; x-z) \Gamma_{2\lambda/3}(s; w-y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \right).
\end{aligned}$$

Applying (i) and (ii) in Lemma 3.3, 3.4 and 3.5, we first have for  $|x-y| \leq t^{1/2}$ , and for  $\{|x-y| > t^{1/2}\} \cap U_{x,y}$ ,

$$\int_0^t \int_{U_{x,y}} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \lesssim p_{2\lambda/3}(t, x, y) N_{F_1}^{\alpha, \lambda/3}(t).$$

This establishes (3.9). For  $|x-y| > t^{1/2}$  and  $(z, w) \in V_{x,y}$ , we apply (iii) in Lemma 3.3, 3.4 and 3.5 to deduce

$$\int_0^t \int_{V_{x,y}} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \lesssim \eta(t; x-y) \|F_1\|_\infty.$$

Hence inequality (3.8) holds.  $\square$

**Lemma 3.7.** *There is a constant  $C > 0$  so that for every  $t \in (0, 2)$  and  $x, y \in \mathbb{R}^d$ ,*

$$\int_{\mathbb{R}^d} p_{2\lambda/3}(t/2, x, z) p_{2\lambda/3}(t/2, z, y) dz \leq C p_{2\lambda/3}(t, x, y). \quad (3.10)$$

**Proof.** It follows from the 3P inequality for  $\eta$  in Lemma 3.1, we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} p_{2\lambda/3}(t/2, x, z) p_{2\lambda/3}(t/2, z, y) dz \\
& \leq \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; x-z) \Gamma_{2\lambda/3}(t/2; z-y) dz + \int_{\mathbb{R}^d} \eta(t/2; x-z) \eta(t/2; z-y) dz \\
& \quad + \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; x-z) \eta(t/2; z-y) dz + \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; z-y) \eta(t/2; x-z) dz \\
& \lesssim \Gamma_{2\lambda/3}(t; x-y) + \eta(t; x-y) + \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; x-z) \eta(t/2; z-y) dz \\
& \quad + \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; z-y) \eta(t/2; x-z) dz,
\end{aligned}$$

for the second to the last term, when  $|x-z| \geq \sqrt{2}|x-y|/2$ ,  $\Gamma_{2\lambda/3}(t/2; x-z) \leq 2^{d/2} \Gamma_{2\lambda/3}(t; x-y)$ ; when  $|x-z| < \sqrt{2}|x-y|/2$ , then  $|y-z| \geq |x-y| - |x-z| \geq \left(1 - \frac{\sqrt{2}}{2}\right) |x-y|$ , we have  $\eta(t/2; z-y) \leq (2/(2-\sqrt{2}))^{d+\alpha} \eta(t; x-y)$ . Thus

$$\int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; x-z) \eta(t/2; z-y) dz \lesssim p_{2\lambda/3}(t, x, y).$$

With similar discussion for the last term, we conclude that (3.10) holds.  $\square$

## 4 Heat Kernel Estimates

In the study of non-local Feynman-Kac perturbation, it is convenient to use Stieltjes exponential rather than the standard exponential. Recall that if  $K_t$  is a right continuous function with left limits on  $\mathbb{R}_+$  with  $K_0 = 1$  and  $\Delta K_t := K_t - K_{t-} > -1$  for every  $t > 0$ , and if  $K_t$  is of finite variation on each compact time interval, then the Stieltjes exponential  $\text{Exp}(K)_t$  of  $K_t$  is the unique solution  $Z_t$  of

$$Z_t = 1 + \int_{(0,t]} Z_{s-} dK_s, \quad t > 0.$$

It is known that

$$\text{Exp}(K)_t = e^{K_t^c} \prod_{0 < s \leq t} (1 + \Delta K_s),$$

where  $K_t^c$  denotes the continuous part of  $K_t$ . The above formula gives a one-to-one correspondence between Stieltjes exponential and the natural exponential. The reason of  $\text{Exp}(K)_t$  being called the *Stieltjes* exponential of  $K_t$  is that, by [16, p. 184],  $\text{Exp}(K)_t$  can be expressed as the following infinite sum of Lebesgue-Stieltjes integrals:

$$\text{Exp}(K)_t = 1 + \sum_{n=1}^{\infty} \int_{[0,t]} dK_{s_n} \int_{[0,s_n)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1}. \quad (4.1)$$

The advantage of using the Stieltjes exponential  $\text{Exp}(K)_t$  over the usual exponential  $\text{Exp}(K_t)$  is the identity (4.1), which allows one to apply the Markov property of  $X$ .

### 4.1 Upper bound estimate

Throughout this subsection,  $\mu \in \mathbf{K}_\alpha$  and  $F$  is a measurable function so that  $F_1 := e^F - 1 \in \mathbf{J}_\alpha$ . We will adopt the approach of [9] to construct and derive its upper bound estimate for the heat kernel of the non-local Feynman-Kac semigroup. Define

$$N_{\mu, F_1}^{\alpha, \lambda}(t) := N_\mu^{\alpha, \lambda}(t) + N_{F_1}^{\alpha, \lambda}(t)$$

and let

$$K_t := A_t^\mu + \sum_{s \leq t} F_1(X_{s-}, X_s). \quad (4.2)$$

Then  $\exp(A_t^\mu + \sum_{s \leq t} F_1(X_{s-}, X_s)) = \text{Exp}(K)_t$ . So it follows from (4.1) that

$$T_t^{\mu, F} f(x) = P_t f(x) + \mathbb{E}_x \left[ f(X_t) \sum_{n=1}^{\infty} \int_{[0,t]} dK_{s_n} \int_{[0,s_n)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1} \right]. \quad (4.3)$$

In view of Theorem 3.2 and Theorem 3.6, we can interchange the order of the expectation and the unfinite sum (see the proof of Theorem 4.3 for details). Using the Markov property of  $X$  and

setting  $h_1(s) := 1$ ,  $h_{n-1}(s) := \int_{[0,s)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1}$ , we have

$$\begin{aligned}
T_t^{\mu,F} f(x) &= P_t f(x) + \sum_{n=1}^{\infty} \mathbb{E}_x \left[ f(X_t) \int_{[0,t]} dK_{s_n} \int_{[0,s_n)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1} \right] \\
&= P_t f(x) + \sum_{n=1}^{\infty} \mathbb{E}_x \left[ \int_{[0,t]} P_{t-s_n} f(X_{s_n}) dK_{s_n} \int_{[0,s_n)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1} \right] \\
&= P_t f(x) + \sum_{n=1}^{\infty} \mathbb{E}_x \left[ \int_{[0,t]} \left( \int_{[0,s_n)} P_{t-s_n} f(X_{s_n}) h_{n-1}(s_{n-1}) dK_{s_{n-1}} \right) dK_{s_n} \right] \\
&= P_t f(x) + \mathbb{E}_x \left[ \int_{[0,t)} \left( \mathbb{E}_{X_{s_{n-1}}} \left[ \int_{(0,t-s_{n-1}]} P_{t-s_{n-1}-r} f(X_r) dK_r \right] \right. \right. \\
&\quad \left. \left. \times \int_{[0,s_{n-1})} dK_{s_{n-2}} \cdots \int_{[0,s_2)} dK_{s_1} \right) dK_{s_{n-1}} \right]. \tag{4.4}
\end{aligned}$$

For any bounded measurable  $g \geq 0$  on  $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ , by Lévy system of  $X$  in (3.6)-(3.7),

$$\begin{aligned}
\mathbb{E}_x \left[ \int_{(0,s]} g(s-r, X_r) dK_r \right] &= \mathbb{E}_x \left[ \int_{(0,s]} g(s-r, X_r) dA_r^\mu + \sum_{r \leq s} g(s-r, X_r) F_1(X_{r-}, X_r) \right] \\
&= \int_0^s \int_{\mathbb{R}^d} p(r, x, y) g(s-r, y) \mu(dy) dr \\
&\quad + \mathbb{E}_x \left[ \int_0^s \left( \int_{\mathbb{R}^d} g(s-r, y) F_1(X_r, y) \frac{c(X_r, y)}{|X_r - y|^{d+\alpha}} dy \right) dr \right] \\
&= \int_0^s \int_{\mathbb{R}^d} p(r, x, y) g(s-r, y) \mu(dy) dr \\
&\quad + \int_0^s \int_{\mathbb{R}^d} p(r, x, y) \left( \int_{\mathbb{R}^d} g(s-r, y) F_1(z, y) \frac{c(z, y)}{|z - y|^{d+\alpha}} dy \right) dz dr. \tag{4.5}
\end{aligned}$$

Define  $p^{(0)}(t, x, y) := p(t, x, y)$ , and for  $k \geq 1$ ,

$$\begin{aligned}
p^{(k)}(t, x, y) &:= \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) p^{(k-1)}(s, z, y) \mu(dz) \right) ds \\
&\quad + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) p^{(k-1)}(s, w, y) \frac{c(z, w) F_1(z, w)}{|z - w|^{d+\alpha}} dz dw \right) ds. \tag{4.6}
\end{aligned}$$

Let

$$q(t, x, y) := \sum_{n=0}^{\infty} p^{(n)}(t, x, y), \tag{4.7}$$

which will be shown in the proof of Theorem 4.3 to be absolutely convergent under the assumption of  $\mu \in \mathbf{K}_\alpha$  and  $F_1 \in \mathbf{J}_\alpha$ . Then it follows from (4.5) and (4.6) that

$$T_t^{\mu,F} f(x) = \int_{\mathbb{R}^d} q(t, x, y) f(y) dy. \tag{4.8}$$

So  $q(t, x, y)$  is the heat kernel for the Feynman-Kac semigroup  $\{T_t^{\mu,F}; t \geq 0\}$ . We will derive upper bound estimate on  $q(t, x, y)$  by estimating each  $p^{(k)}(t, x, y)$ .

**Lemma 4.1.** *There are constants  $C_0 \geq 1$  and  $M \geq 1$  such that for every  $k \geq 0$  and  $(t, x) \in (0, 1] \times \mathbb{R}^d$ ,*

$$\int_{\mathbb{R}^d} |p^{(k)}(t, x, y)| dy \leq C_0 \left( MN_{\mu, F_1}^{\alpha, \lambda}(t) \right)^k. \quad (4.9)$$

**Proof.** We prove this lemma by induction. When  $k = 0$ , by Lemma 2.1, we have the inequality hold naturally. Suppose (4.9) is true for  $k - 1$ . Then by (4.6),

$$\begin{aligned} \int_{\mathbb{R}^d} |p^{(k)}(t, x, y)| dy &\leq \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) \left( \int_{\mathbb{R}^d} p^{(k-1)}(s, z, y) dy \right) |\mu|(dz) \right) ds \\ &\quad + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} \left( \int_{\mathbb{R}^d} p^{(k-1)}(s, w, y) dy \right) dz dw \right) ds \\ &\leq C_0 \left( MN_{\mu, F_1}^{\alpha, \lambda}(t) \right)^{k-1} \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) |\mu|(dz) ds \\ &\quad + C_0 \left( MN_{\mu, F_1}^{\alpha, \lambda}(t) \right)^{k-1} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ &\leq C_0 C (1 + \|c\|_\infty) M^{k-1} \left( N_{\mu, F_1}^{\alpha, \lambda}(t) \right)^k \leq C_0 \left( MN_{\mu, F_1}^{\alpha, \lambda}(t) \right)^k, \end{aligned}$$

if we increase the value of  $M$  if necessary so that  $M \geq C(1 + \|c\|_\infty)$ . Here  $C \geq 1$  is the constant in Lemma 2.1. The lemma is proved.  $\square$

**Lemma 4.2.** *For any  $k \geq 0$  and  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ ,*

$$|p^{(k)}(t, x, y)| \leq C p_{2\lambda/3}(t, x, y) \left( (MN_{\mu, F_1}^{\alpha, \lambda/3}(t))^k + k \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3}(t))^{k-1} \right), \quad (4.10)$$

where  $C \geq 1$  and  $M \geq 1$  are the constants in Lemma 2.1 and Lemma 4.1, respectively.

**Proof.** Inequality holds trivially for  $k = 0$ . Suppose it is true for  $k - 1 \geq 0$ , then if  $|x - y| \leq t^{1/2}$ , using the induction hypothesis and applying Theorem 3.2 and Theorem 3.6,

$$\begin{aligned} |p^{(k)}(t, x, y)| &\leq \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) |p^{(k-1)}(s, z, y)| |\mu|(dz) \right) ds \\ &\quad + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} |p^{(k-1)}(s, w, y)| dz dw \right) ds \\ &\leq C \left( (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1} + (k-1) \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-2} \right) \\ &\quad \times \left( \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) \right) ds \right. \\ &\quad \left. + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw \right) ds \right) \\ &\leq C p_{2\lambda/3}(t, x, y) \left( (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1} + (k-1) \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-2} \right) MN_{\mu, F_1}^{\alpha, \lambda/3}(t). \end{aligned}$$

If  $|x - y| > t^{1/2}$ , we have

$$\begin{aligned}
|p^{(k)}(t, x, y)| &\leq \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) |p^{(k-1)}(s, z, y)| |\mu|(dz) \right) ds \\
&\quad + \int_0^t \left( \int_{U_{x,y}} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} |p^{(k-1)}(s, w, y)| dz dw \right) ds \\
&\quad + \int_0^t \left( \int_{V_{x,y}} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} |p^{(k-1)}(s, w, y)| dz dw \right) ds \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

Applying Theorem 3.2 to  $J_1$  and Theorem 3.6 to  $J_2$ ,

$$\begin{aligned}
J_1 + J_2 &\leq C \left( (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1} + (k-1) \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-2} \right) \\
&\quad \times \left( \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(t, x, y) |\mu|(dz) ds \right. \\
&\quad \left. + \int_0^t \left( \int_{U_{x,y}} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw \right) ds \right) \\
&\leq C p_{2\lambda/3}(t, x, y) \left( (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1} + (k-1) \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-2} \right) MN_{\mu, F_1}^{\alpha, \lambda/3}.
\end{aligned}$$

For  $J_3$ , use the fact that  $|z - w| \geq 2^{-1}|x - y|$  and Lemma 4.1,

$$\begin{aligned}
J_3 &\leq \frac{2^{d+\alpha} \|F_1\|_\infty}{|x - y|^{d+\alpha}} \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) |p^{(k-1)}(s, w, y)| dz dw \right) ds \\
&\leq 2^{d+\alpha} \|F_1\|_\infty \frac{t}{|x - y|^{d+\alpha}} C_0^2 (MN_{\mu, F_1}^{\alpha, \lambda})^{k-1} \\
&\leq M \|F_1\|_\infty p_{2\lambda/3}(t, x, y) (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1}.
\end{aligned}$$

This completes the proof.  $\square$

The following result gives the existence and the desired upper bound estimates of the heat kernel for the non-local Feynman-Kac semigroup  $\{T_t^{\mu, F}; t \geq 0\}$ , as stated in Theorem 2.3.

**Theorem 4.3.** *The series  $\sum_{k=0}^\infty p^{(k)}(t, x, y)$  converges absolutely to a jointly continuous function  $q(t, x, y)$  on  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ . The function  $q(t, x, y)$  is the integral kernel (or, heat kernel) for the Feynman-Kac semigroup  $\{T_t^{\mu, F}; t \geq 0\}$ , and there exists a constant  $K$  depending on  $d, \alpha, \|F_1\|_\infty$  and the constants  $C$  and  $\lambda := c_4$  in Lemma 2.1 such that*

$$q(t, x, y) \leq e^{Kt} p_{2\lambda/3}(t, x, y) \quad \text{for every } (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \quad (4.11)$$

**Proof.** Let  $\hat{p}^{(k)}(t, x, y)$  be defined as in (4.6) but with  $|\mu|$  and  $|F_1|$  in place of  $\mu$  and  $F_1$ ; that is,  $\hat{p}^{(0)}(t, x, y) = p(t, x, y)$ , and for  $k \geq 1$ ,

$$\begin{aligned}
\hat{p}^{(k)}(t, x, y) &:= \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) \hat{p}^{(k-1)}(s, z, y) |\mu|(dz) \right) ds \\
&\quad + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) \hat{p}^{(k-1)}(s, w, y) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw \right) ds.
\end{aligned} \quad (4.12)$$



Clearly,  $|p^{(k)}(t, x, y)| \leq \widehat{p}^{(k)}(t, x, y)$  and by the proof of Lemma 4.2, there is a constant  $0 < t_1 \leq 1$  so that such that  $N_{\mu, F_1}^{\alpha, \lambda/3}(t_1) \leq (2M)^{-1}$  and that

$$\begin{aligned}
\widehat{q}(t, x, y) &:= \sum_{k=0}^{\infty} \widehat{p}^{(k)}(t, x, y) \\
&\leq Cp_{2\lambda/3}(t, x, y) + Cp_{2\lambda/3}(t, x, y) \sum_{k=1}^{\infty} \left( (MN_{\mu, F_1}^{\alpha, \lambda/3}(t))^k + k\|F_1\|_{\infty} M (MN_{\mu, F_1}^{\alpha, \lambda/3}(t))^{k-1} \right) \\
&\leq Cp_{2\lambda/3}(t, x, y) + Cp_{2\lambda/3}(t, x, y) (1 + 4\|F_1\|_{\infty} M) \\
&\leq C(2 + 4\|F_1\|_{\infty} M) p_{2\lambda/3}(t, x, y) =: \gamma_1 p_{2\lambda/3}(t, x, y).
\end{aligned} \tag{4.13}$$

This in particular implies that  $\widehat{q}(t, x, y)$  is jointly continuous on  $(0, t_1] \times \mathbb{R}^d \times \mathbb{R}^d$ . Repeating the procedure (4.3), (4.4) and (4.5) with  $|\mu|$ ,  $|F_1|$  in place of  $\mu$ ,  $F_1$ . and by Fubini's theorem, we have for any bounded function  $f \geq 0$  on  $\mathbb{R}^d$  and  $t \in (0, t_1]$ ,

$$T_t f(x) := \mathbb{E}_x \left[ f(X_t) \text{Exp} \left( A^{|\mu|} + \sum_{s \leq \cdot} |F_1|(X_{s-}, X_s) \right)_t \right] = \int_{\mathbb{R}^d} \widehat{q}(t, x, y) f(y) dy. \tag{4.14}$$

Note that  $T_t \circ T_s = T_{t+s}$  for any  $t, s \geq 0$ . Extend the definition of  $\widehat{q}(t, x, y)$  to  $(0, 2t_1] \times \mathbb{R}^d \times \mathbb{R}^d$  by

$$\widehat{q}(t+s, x, y) = \int_{\mathbb{R}^d} \widehat{q}(t, x, z) \widehat{q}(s, z, y) dz$$

for  $s, t \in (0, t_1]$ . The above is well defined and, in view of (4.13) and Lemma 3.7,  $\widehat{q}(t, x, y)$  is jointly continuous on  $[0, 2t_1] \times \mathbb{R}^d \times \mathbb{R}^d$  and there is constant  $\gamma_2$  so that  $\widehat{p}(t, x, y) \leq \gamma_2 p_{2\lambda/3}(t, x, y)$  on  $(0, 2t_1] \times \mathbb{R}^d \times \mathbb{R}^d$ . Clearly,

$$T_t f(x) = \int_{\mathbb{R}^d} \widehat{q}(t, x, y) f(y) dy$$

for every  $f \geq 0$  on  $\mathbb{R}^d$  and  $(t, x) \in (0, 2t_1] \times \mathbb{R}^d$ . Repeat the above procedure, we can extend  $\widehat{q}(t, x, y)$  to be a jointly continuous function on  $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$  so that (4.14) holds for every  $f \geq 0$  on  $\mathbb{R}^d$  and  $(t, x) \in (0, \infty) \times \mathbb{R}^d$ , and that there exists a constant  $K > 0$  depending on  $d, \alpha, C, \lambda, C_0, \|F_1\|_{\infty}, M$  so that for any  $t > 0$  and  $x, y \in \mathbb{R}^d$

$$\widehat{q}(t, x, y) \leq e^{Kt} p_{2\lambda/3}(t, x, y).$$

This proves the theorem as  $q(t, x, y) \leq \widehat{q}(t, x, y)$ .  $\square$

## 4.2 Lower bound estimate

In this subsection, we assume  $\mu \in \mathbf{K}_{\alpha}$  and  $F \in \mathbf{J}_{\alpha}$ . Clearly,  $F_1 := e^F - 1 \in \mathbf{J}_{\alpha}$ . Due to the presence of the Gaussian component in (1.3), the approach in [9] of obtaining lower bound estimates for  $q(t, x, y)$  is not applicable here. We will employ a probabilistic approach from [13, 14] to get the desired lower bound estimates.

Let  $\widetilde{p}^{(1)}(t, x, y)$  be defined as in (4.12) but with  $|F|$  in place of  $|F_1|$ . Thus by (4.13), there is a constant  $\gamma > 0$  so that

$$\widetilde{p}^{(1)}(t, x, y) \leq \gamma p_{2\lambda/3}(t, x, y) \quad \text{for } (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d.$$

In particular, there is a constant  $K_1 > 0$  so that  $\tilde{p}^1(t, x, y) \leq K_1 t^{-d/2}$  for  $t \in (0, 1]$  and  $|x - y| \leq \sqrt{t}$ . On the other hand, it follows from (2.1) that there exists a constant  $\tilde{C} \geq 1$  so that

$$\tilde{C}^{-1} t^{-d/2} \leq p(t, x, y) \leq \tilde{C} t^{-d/2} \quad \text{for every } t \in (0, 1] \text{ and } |x - y| \leq \sqrt{t}.$$

Let  $k \geq 2$  be an integer so that  $k \geq 2K_1 \tilde{C}$ . Then for every  $t \in (0, 1]$  and  $x, y \in \mathbb{R}^d$  with  $|x - y| \leq \sqrt{t}$ ,

$$p(t, x, y) - \frac{1}{k} \tilde{p}^1(t, x, y) \geq \frac{1}{2\tilde{C}} t^{-d/2} \geq \frac{1}{2\tilde{C}^2} p(t, x, y). \quad (4.15)$$

Note that

$$\mathbb{E}_x \left[ A_t^{|\mu|, |F|} f(X_t) \right] = \int_{\mathbb{R}^d} \tilde{p}^{(1)}(t, x, y) f(y) dy.$$

Using the elementary inequality that

$$1 - A_t^{|\mu|, |F|} / k \leq \exp(-A_t^{|\mu|, |F|} / k) \leq \exp(A_t^{\mu, F} / k),$$

we have for any ball  $B(x, r)$  centered at  $x$  with radius  $r$  and any  $(t, y) \in (0, 1] \times \mathbb{R}^d$ ,

$$\frac{1}{|B(x, r)|} \mathbb{E}_y \left[ (1 - A_t^{|\mu|, |F|} / k) \mathbf{1}_{B(x, r)}(X_t) \right] \leq \frac{1}{|B(x, r)|} \mathbb{E}_y \left[ \exp(A_t^{\mu, F} / k) \mathbf{1}_{B(x, r)}(X_t) \right].$$

Hence by (4.15) and Hölder's inequality, we have for  $0 < t \leq 1$  and  $|x - y| \leq \sqrt{t}$ ,

$$\begin{aligned} \frac{1}{2\tilde{C}^2} \frac{1}{|B(x, r)|} \mathbb{E}_y [\mathbf{1}_{B(x, r)}(X_t)] &\leq \frac{1}{|B(x, r)|} \mathbb{E}_y [\exp(A_t^{\mu, F} / k) \mathbf{1}_{B(x, r)}(X_t)] \\ &\leq \left( \frac{1}{|B(x, r)|} \mathbb{E}_y [\exp(A_t^{\mu, F}) \mathbf{1}_{B(x, r)}(X_t)] \right)^{1/k} \left( \frac{1}{|B(x, r)|} \mathbb{E}_y [\mathbf{1}_{B(x, r)}(X_t)] \right)^{1-1/k}. \end{aligned}$$

Thus

$$\frac{1}{2^k \tilde{C}^{2k}} \frac{1}{|B(x, r)|} \mathbb{E}_y [\mathbf{1}_{B(x, r)}(X_t)] \leq \frac{1}{|B(x, r)|} \mathbb{E}_y [\text{Exp}(A_t^{\mu, F}) \mathbf{1}_{B(x, r)}(X_t)].$$

By taking  $r \downarrow 0$ , we conclude from above as well as Lemma 2.1 that

$$q(t, x, y) \geq 2^{-k} \tilde{C}^{-2k} p(t, x, y) \gtrsim t^{-d/2} \quad \text{for every } t \in (0, 1] \text{ and } |x - y| \leq \sqrt{t}. \quad (4.16)$$

By a standard chaining argument (see, e.g., [17]), it follows that there exist constants  $K_2, \lambda_1 > 0$  so that

$$q(t, x, y) \geq K_2 \Gamma_{\lambda_1}(t; x - y) \quad \text{for } (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (4.17)$$

To get the jumping component in the lower bound estimate for  $q(t, x, y)$ , we consider a sub-Markovian semigroup  $\{Q_t; t \geq 0\}$  defined by

$$Q_t f(x) := \mathbb{E}_x \left[ \text{Exp} \left( -A_t^{|\mu|} - \sum_{s \leq t} |F|(X_{s-}, X_s) \right) f(X_t) \right].$$

Since  $|\mu| \in \mathbf{K}_\alpha$  and  $|F| \in \mathbf{J}_\alpha$ , we know that  $\{Q_t; t \geq 0\}$  has a jointly continuous transition kernel  $\bar{p}(t, x, y)$ . Clearly,  $q(t, x, y) \geq \bar{p}(t, x, y)$  for every  $t > 0$  and  $x, y \in \mathbb{R}^d$ . Since  $\{Q_t; t \geq 0\}$  forms a Feller semigroup, there exists a Feller process  $Y = \{Y_t, \mathbb{P}_x, x \in \mathbb{R}^d, \zeta^Y\}$  such that  $Q_t f(x) = \mathbb{E}_x[f(Y_t)]$ . We will derive a lower bound estimate on  $q(t, x, y)$  through the Feller process  $Y$ .

It follows from the definition of  $\mu \in \mathbf{K}_\alpha$  and  $F \in \mathbf{J}_\alpha$  that  $\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ A_1^{|\mu|, |F|} \right] < \infty$ . Thus by Jensen's inequality,

$$\inf_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ \text{Exp}(A^{-|\mu|, -|F|})_1 \right] \geq \text{Exp} \left( - \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ A_1^{|\mu|, |F|} \right] \right) =: \gamma_0 > 0. \quad (4.18)$$

Let  $\eta$  be the random time whose distribution is determined by  $\mathbb{P}_x(\zeta > t) = \mathbb{E}_x [\text{Exp}(A^{-|\mu|, -|F|})_t]$ . We can couple the processes  $X$  and  $Y$  in such a way that on  $\{\eta > t\}$ ,  $Y_s = X_s$  for every  $s \leq t$ .

We define first hitting time and exit time of a Borel set  $D \subset \mathbb{R}^d$  by  $X_t$  and  $Y_t$  as follows;

$$\begin{aligned} \sigma_D^X &:= \inf\{s \geq 0, X_s \in D\}, & \tau_D^X &:= \inf\{s \geq 0, X_s \notin D\}; \\ \sigma_D^Y &:= \inf\{s \geq 0, Y_s \in D\}, & \tau_D^Y &:= \inf\{s \geq 0, Y_s \notin D\}. \end{aligned}$$

**Lemma 4.4.** *Let  $\gamma_0 \in (0, 1)$  be the constant in (4.18). There exists a constant  $\kappa_0 \in (0, 1)$  depending on  $d, C, \lambda, \alpha, R, \gamma_0$  such that for any  $0 < r \leq 1$ ,*

$$\sup_{x \in \mathbb{R}^d} \mathbb{P}_x \left( \tau_{B(x, r)}^X \leq \kappa_0 r^2 \right) \leq \gamma_0/2. \quad (4.19)$$

Consequently, for every  $x \in \mathbb{R}^d$  and  $r \in (0, 1]$ ,

$$\mathbb{P}_x \left( \tau_{B(x, r)}^Y > \kappa_0 r^2 \right) \geq \gamma_0/2. \quad (4.20)$$

**Proof.** First note that by (1.3), for every  $s \leq t \leq 1$ ,  $x \in \mathbb{R}^d$  and  $r > 0$ ,

$$\int_{B(x, r)^c} p(s, x, y) dy \leq c \left( \int_{B(0, r/2\sqrt{s})^c} e^{-c_4|z|^2} dz + sr^{-\alpha} \right) \leq c \left( e^{-c_4 r^2/8t} + tr^{-\alpha} \right).$$

Let  $\kappa_0 > 0$  sufficiently small so that  $c(e^{-c_4/8\kappa_0} + \kappa_0) < \gamma_0/4$ . Then by taking  $t = \kappa_0 r^2$ , we have from the above that for every  $x \in \mathbb{R}^d$  and  $r \in (0, 1]$ ,

$$\sup_{s \in (0, \kappa_0 r^2]} \int_{B(x, r/2)^c} p(s, x, y) dy \leq \gamma_0/4. \quad (4.21)$$

For simplicity, denote  $\tau_{B(x, r)}^X$  by  $\tau$ . We have by the strong Markov property of  $X_t$  and (4.21) that

$$\begin{aligned} \mathbb{P}_x(\tau \leq \kappa_0 r^2) &\leq \mathbb{P}_x(\tau \leq \kappa_0 r^2; X_{\kappa_0 r^2} \in B(x, r/2)) + \mathbb{P}_x(X_{\kappa_0 r^2} \notin B(x, r/2)) \\ &\leq \mathbb{P}_x(\mathbb{P}_{X_\tau}(|X_{\kappa_0 r^2 - \tau} - X_0| \geq r/2); \tau \leq \kappa_0 r^2) + \frac{\gamma_0}{4} \\ &\leq \frac{\gamma_0}{4} + \frac{\gamma_0}{4} = \frac{\gamma_0}{2}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{P}_x \left( \tau_{B(x, r)}^Y > \kappa_0 r^2 \right) &\geq \mathbb{P}_x \left( \eta > \kappa_0 r^2 \text{ and } \tau_{B(x, r)}^X > \kappa_0 r^2 \right) \\ &\geq \mathbb{P}_x(\eta > \kappa_0 r^2) - \mathbb{P}_x \left( \tau_{B(x, r)}^X \leq \kappa_0 r^2 \right) \geq \gamma_0/2, \end{aligned}$$

where in the last inequality, we used (4.18).  $\square$

**Lemma 4.5.** *Let  $0 < \kappa_0 < 1$  be the constant in Lemma 4.4. There exists a constant  $\gamma_1 > 0$  so that for any  $r > 0$  and  $x_0, y_0 \in \mathbb{R}^d$  with  $|y_0 - x_0| \geq 3r$ ,*

$$\mathbb{P}_{x_0} \left( \sigma_{B(y_0, r)}^Y \leq \kappa_0 r^2 \right) \geq \gamma_1 \frac{r^{d+2}}{|y_0 - x_0|^{d+\alpha}}. \quad (4.22)$$

**Proof.** Define  $f(x, y) = 1_{B(x_0, r)}(x)1_{B(y_0, r)}(y)$ . Then

$$M_t := \sum_{s \leq t \wedge (\kappa_0 r^2)} f(X_{s-}, X_s) - \int_0^{t \wedge (\kappa_0 r^2)} f(X_s, y) N(X_s, dy) ds, \quad t \geq 0,$$

is a martingale additive functional of  $X$  that is uniformly integrable under  $\mathbb{P}_x$  for every  $x \in \mathbb{R}^d$ . Let

$$A_t := A_t^{-|\mu|, |F|} = -A_t^{|\mu|} - \sum_{s \leq t} F(X_{s-}, X_s), \quad t \geq 0,$$

which is a non-increasing additive functional of  $X$ . By stochastic integration by parts formula,

$$e^{A_t} M_t = \int_0^t e^{A_{s-}} dM_s + \int_0^t M_{s-} de^{A_s} + \sum_{s \leq t} (e^{A_s} - e^{A_{s-}})(M_s - M_{s-}).$$

For  $\tau := \tau_{B(x_0, r)}^X \wedge (\kappa_0 r^2)$ ,  $M_t = - \int_0^{t \wedge (\kappa_0 r^2)} f(X_s, y) N(X_s, dy) ds \leq 0$  for  $t \in [0, \tau]$ . It follows that

$$e^{A_\tau} M_\tau \geq \int_0^\tau e^{A_{s-}} dM_s + (e^{A_\tau} - e^{A_{\tau-}}) M_\tau.$$

Thus

$$\mathbb{E}_{x_0} [e^{A_{\tau-}} M_\tau] \geq \mathbb{E}_{x_0} \int_0^\tau e^{A_{s-}} dM_s = 0.$$

This together with (3.7) implies that

$$\begin{aligned} \mathbb{E}_{x_0} [e^{A_{\tau-}} 1_{B(y_0, r)}(X_\tau)] &\geq \mathbb{E}_{x_0} \left[ e^{A_{\tau-}} \int_0^{\tau \wedge (\kappa_0 r^2)} \int_{B(y_0, r)} N(X_s, dy) ds \right] \\ &\geq \frac{c \kappa_0 r^2 r^d}{|x_0 - y_0|^{d+\alpha}} \mathbb{E}_{x_0} \left[ e^{A_{\kappa_0 r^2}}; \tau_{B(x_0, r)}^X \geq \kappa_0 r^2 \right] \\ &= \frac{c \kappa_0 r^{d+2}}{|x_0 - y_0|^{d+\alpha}} \mathbb{P}_{x_0} \left( \tau_{B(x_0, r)}^Y \geq \kappa_0 r^2 \right) \\ &\geq \frac{c \gamma_0 \kappa_0 r^{d+2}}{2|x_0 - y_0|^{d+\alpha}}, \end{aligned}$$

where the last inequality is due to (4.20). Consequently,

$$\begin{aligned} \mathbb{P}_{x_0} \left( \sigma_{B(y_0, r)}^Y \leq \kappa_0 r^2 \right) &\geq \mathbb{P}_{x_0} \left( \tau_{B(x_0, r)}^Y \leq \kappa_0 r^2 \text{ and } Y_{\tau_{B(x_0, r)}^Y} \in B(y_0, r) \right) \\ &= \mathbb{E}_{x_0} [e^{A_\tau} 1_{B(y_0, r)}(X_\tau)] \\ &\geq e^{-\|F\|_\infty} \mathbb{E}_{x_0} [e^{A_{\tau-}} 1_{B(y_0, r)}(X_\tau)] \\ &\geq \frac{c e^{-\|F\|_\infty} \gamma_0 \kappa_0 r^{d+2}}{2|x_0 - y_0|^{d+\alpha}}. \end{aligned}$$

The lemma is proved.  $\square$

We now derive lower bound heat kernel estimate for the heat kernel  $q(t, x, y)$  of the Feynman-Kac semigroup  $\{T_t^{\mu, F}; t \geq 0\}$ .

**Theorem 4.6.** *Suppose  $\mu \in \mathbf{K}_\alpha$  and  $F$  is a measurable function in  $\mathbf{J}_\alpha$ . Then there exist positive constants  $\tilde{K} \geq 1$  and  $\lambda_1 > 0$  depending on  $(d, \alpha, \lambda, N_{\alpha, F}^{\alpha, \lambda/3}, \|F\|_\infty)$  and the constants in (1.3) such that*

$$\tilde{K}^{-1} p_{\lambda_1}(t, x, y) \leq q(t, x, y) \leq \tilde{K} p_{2\lambda/3}(t, x, y) \quad (4.23)$$

for  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ .

**Proof.** The upper bound estimates follows from Theorem 4.3 so it remains to establish the lower bound estimate for  $q(t, x, y)$ . If  $|x - y| \leq \sqrt{t}$ , the desired lower bound heat kernel estimate follows from (4.17). So it suffices to consider the case that  $|x - y| > \sqrt{t}$ . Set  $r = \sqrt{t}/3$ . It follows from Lemma 4.4 and Lemma 4.5 that

$$\begin{aligned} \mathbb{P}_x(Y_{2\kappa_0 r^2} \in B(y, 2r)) &\geq \mathbb{P}_x\left(\sigma_{B(y, r)}^Y < \kappa_0 r^2; \sup_{s \in [\sigma, \sigma + \kappa_0 r^2]} |Y_s - Y_\sigma| < r\right) \\ &= \mathbb{E}_x\left[\mathbb{P}_{Y_\sigma}\left(\sup_{s \in [\sigma, \sigma + \kappa_0 r^2]} |Y_s - Y_\sigma| < r\right); \sigma < \kappa_0 r^2\right] \\ &\geq \mathbb{P}_x\left(\tau_{B(x, r)}^Y > \kappa_0 r^2\right) \mathbb{P}_x\left(\sigma_{B(y, r)}^Y < \kappa_0 r^2\right) \\ &\geq \frac{\gamma_0 \gamma_1}{2} \frac{r^{d+2}}{|x - y|^{d+\alpha}}. \end{aligned}$$

Thus

$$\int_{B(y, 2r)} q(2\kappa_0 r^2, x, z) dz \geq \mathbb{P}_x(Y_{2\kappa_0 r^2} \in B(y, 2r)) \geq \frac{\gamma_0 \gamma_1 r^{d+2}}{2|x - y|^{d+\alpha}}.$$

Since  $|y - z| < 2r < \sqrt{t - 2\kappa_0 r^2}$ , one has by (4.16) that

$$\begin{aligned} q(t, x, y) &\geq \int_{B(y, 2r)} q(2\kappa_0 r^2, x, z) q(t - 2\kappa_0 r^2, z, y) dz \\ &\geq \inf_{z \in B(y, 2r)} q(t - 2\kappa_0 r^2, y, z) \frac{\gamma_0 \gamma_1 r^{d+2}}{2|x - y|^{d+\alpha}} \\ &\geq K_2 e^{-\lambda_1 t^{-d/2}} \frac{\gamma_0 \gamma_1 r^{d+2}}{2|x - y|^{d+\alpha}} \\ &\geq K_2 e^{-\lambda_1} \frac{\gamma_0 \gamma_1}{2 \cdot 3^{d+2}} \eta(t; x - y). \end{aligned}$$

This together with (4.17) establishes the lower bound estimate for  $q(t, x, y)$  in (4.23).  $\square$

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Department of Mathematics, University of Washington, Seattle, WA 98195, USA

Email: [zqchen@uw.edu](mailto:zqchen@uw.edu)

Email: [lidanw@uw.edu](mailto:lidanw@uw.edu)